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STATIC BLACK HOLES IN BRANS-DICKE THEORY

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled STATIC BLACK HOLES IN BRANS-DICKE THEORY submitted by MARY L. EASWARAN, B.Sc. (Honours) in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The major objective of this thesis is to investigate the conjecture that static black holes in Brans-Dicke theory are precisely the same as in general relativity. With this objective in mind, we first give a rigorous mathematical definition of a black hole, due to Hawking, and then discuss the Israel-Carter conjecture, that a black hole always settles down to a Kerr solution, assuming everything becomes stationary eventually. We then discuss the Brans-Dicke theory, giving a brief summary of the predictions and consequences of this theory. Finally, we prove that the scalar field ϕ of the Brans-Dicke theory must be constant if a black hole occurs in a static space-time. Thus, for the static case, black holes in Brans-Dicke theory are precisely the same as in general relativity.

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Introduction

§1.1 Black Holes

Black holes have been the subject of a great deal of investigation in recent years. Briefly, a black hole is a region from which no future-directed null or timelike curve can escape; it is assumed that singularities in gravitational collapse are hidden in black holes.

It is known that black holes exist in the Schwarzschild and Kerr solutions of general relativity, and there has been a great deal of interest in the question of whether other solutions also yield black holes. It has been conjectured that every black hole settles down to a Kerr solution, assuming that everything becomes stationary eventually. The results of Israel [14], Carter [3,4], Hawking [12], and others lend a great deal of support to this conjecture.

§1.2 Brans-Dicke Theory

Although Einstein's general theory of relativity is widely accepted by physicists, there are valid reasons for postulating rival theories. Perhaps the major reason is the opportunity of testing general relativity which a rival theory gives, as it is difficult to test a prediction

of general relativity unless one has the prediction of another theory for comparison. Many rival theories have been postulated over the years, a major one being the scalar-tensor theory of C. Brans and R. H. Dicke [2] which first appeared in 1961.

This theory was originally motivated by an attempt to incorporate Dicke's interpretation of Mach's principle into general relativity. Dicke introduced a scalar field ϕ, which has as its source the matter distribution in space. The locally measured gravitational constant G varies as the inverse of the scalar field ϕ. Note, therefore, that the gravitational "constant" of Einstein's theory is not constant in Brans-Dicke theory.

Up to the present time, this theory has not been disproved experimentally, as no differences have been found in qualitative predictions, and the differences in quantitative predictions have been so slight as to be hidden by the error bounds in the present experiments.

§1.3 Black Holes in Brans-Dicke Theory

In the light of the preceding remarks, it would be of interest to study black holes in the framework of the Brans-Dicke theory to discover whether there is any qualitative difference between the two theories concerning this phenomenon.

Chapter II contains a survey of recent mathematical work on black holes, including the precise mathematical definition proposed by Hawking, and a discussion of the results of Israel, Carter, and Hawking concerning stationary black holes.

In Chapter III we will discuss some of the predictions and consequences of Brans-Dicke theory, and the supporting experimental evidence.

In Chapter IV we will prove our result that static black holes in Brans-Dicke theory are Schwarzschild [15]. This result had been expected, and has recently been obtained independently by Hawking [13] and Bekenstein [1].

CHAPTER II

Black Holes

In various recent papers, Penrose [24], Hawking [12], Geroch [11], and others have applied the methods of topology and differential geometry to relativity. This approach allows a rigorous mathematical definition of a black hole to be formulated. For this purpose, then, it is useful to review the pertinent definitions.

§2.1 Definitions

Throughout this discussion, M will be a 4-dimensional time-oriented differentiable manifold with a smooth metric $g_{\alpha\beta}$ of signature (-+++).

If there is a future-directed geodesic path (causal trip) from a to b, whose pieces may be either timelike or null, we write $a<b$ (a causally precedes b), where a path is a broken geodesic. $J^+(a)\equiv\{b|a<b\}$ is called the causal future of a. If, however, we are restricted to piecewise timelike geodesics (trips), then we write $a<<b$ (a chronologically precedes b), and $I^+(a)\equiv$ $\equiv\{b|a<<b\}$ is called the chronological future of a. The pasts are defined similarly. Let $S\subseteq M$. Then the future of S is $I^+[S]\equiv\bigcup_{x\in S}\{I^+(x)\}$. It can be shown [17] that $I^{\pm}[S]$

is open for any S, $\overline{I^+[S]}=\{x|I^+(x)\subseteq S\}$, and, therefore, $\partial I^+[S]=\{x|I^+(x)\subseteq S \text{ and } x\notin S\}$.

A subset S of M is called <u>achronal</u> iff no two points of S are chronologically related.

Define the set $E^+[S]\equiv J^+[S]\setminus I^+[S]$. Then $E^+[S]\subseteq \partial I^+[S]$, and, if S is closed, $E^+[S]$ is the portion of $\partial I^+[S]$ which may be connected to S by null geodesics. A non-empty, closed, achronal set S is said to be <u>future-trapped</u> if $E^+[S]$ is compact.

Let $S\subseteq M$. Define $D^+(S)\equiv\{x\varepsilon M|$ every trip through x with no past endpoint meets S$\}$. Similarly for $D^-(S)$. Then $D(S)\equiv D^+(S)\cup D^-(S)$ is called the <u>domain of dependence</u> of S.

Penrose's concept of an asymptotically simple space-time [24] is the next tool used by Hawking in his definition of a black hole. If a space-time M, with metric ds^2, is extendible, i.e. to a conformal manifold with boundary $\overline{M}\supset M$ (int $\overline{M}=M$, $\partial\overline{M}=\overline{M}\setminus M$) such that there exists a smooth, real-valued function $\Omega(\geqslant 0)$ on $\overline{M}$ and a smooth pseudo-Riemannian metric $d\hat{s}^2$ on $\overline{M}$ (consistent with its conformal structure), such that $ds^2=\Omega^2 d\hat{s}^2$ on M, on $\partial\overline{M}$ we have $\Omega=0$, $\hat{\nabla}_\alpha\Omega\neq 0$, and every null geodesic in M has two endpoints on $\partial\overline{M}$, then M is said to be <u>asymptotically simple</u>. A space-time M is said to be <u>weakly asymptotically simple</u>

if an asymptotically simple M_o exists such that for some open subset K of $\overline{M}_o$, with $\partial M_o \subset K$, the region $M_o \cap K$ (with metric induced from M_o) is isometric with a subset of M. That is, a weakly asymptotically simple space-time posseses the conformal infinity $\mathcal{I} = \partial M_o$ of an asymptotically simple space-time, but it may possess other "infinities" as well. The boundary of M in $\overline{M}$ consists of two null hypersurfaces $\mathcal{I}^+$ and $\mathcal{I}^-$, which each have topology $S^2 \times R^1$ and which represent future and past null infinity respectively. (Weak asymptotic simplicity is, in essentials, the same as Bondi-Sachs asymptotic flatness).

Hawking [12] then says that a weakly asymptotically simple space is (<u>future</u>) <u>asymptotically predictable</u> iff there exists a partial Cauchy surface S such that $\mathcal{I}^+$ lies in the closure in $\overline{M}$ of $D^+(S)$. ($\{I^+[S] \cap I^-[\mathcal{I}^+]\} \subseteq D^+(S)$). A <u>partial Cauchy surface</u> is a space-like surface without edge which does not intersect any non-space-like curve more than once. If $S \subset M$ is achronal and closed, the <u>edge</u> of S is defined by edge $S \equiv \{x \varepsilon S \mid \forall y, z, \ni z << x < y$, there is a trip [zy] not meeting $S\}$.

In an asymptotically predictable space, it can be shown [12] that the future incomplete non-space-like geodesic in $I^+[T]$, where T is a closed, trapped surface, is invisible from $\mathcal{I}^+$. That is, $T \cap I^-[\mathcal{I}^+] = \phi$. Since $I^-[\mathcal{I}^+]$ does not contain T, its boundary, $\partial I^-[\mathcal{I}^+]$, must be non-

empty, and this is the event horizon. It is the boundary of the region from which particles or photons can escape to infinity.

A space is strongly asymptotically predictable if, in addition to being asymptotically predictable, $I^+[S] \cap \partial I^-[\mathscr{I}^+]$ is in $D^+(S)$. In a strongly asymptotically predictable space, one can construct a family $S(t)$ $(t>0)$ of partial Cauchy surfaces in $D^+(S)$ such that

(a) for $t_2>t_1$, $S(t_2) \subset I^+[S(t_1)]$;

(b) each $S(t)$ intersects $\mathscr{I}^+$ in a 2-sphere $A(t)$;

(c) for each $t>0$, $S(t) \cup [\mathscr{I}^+ \cap I^-[A(t)]]$ is a Cauchy surface for $D^+(S)$. [12].

For sufficiently large t, the surfaces $S(t)$ will intersect the event horizon, so $B(t)$ defined as $S(t) \cap \partial I^-[\mathscr{I}^+]$ will be non-empty. A black hole on the surface $S(t)$ is a connected component of $B(t)$.

Several properties of black holes follow quickly from this definition. [12]. As time increases, black holes may merge together, and new black holes may be created by further bodies collapsing, but a black hole can never bifurcate. Also, the surface area of $B_1(t_1)$ and $B_2(t_1)$ on the surface $S(t_1)$ merge to form a single black hole $B_3(t_2)$ on a later surface $S(t_2)$, then the area of $B_3(t_2)$ must be strictly greater than the sum of the areas of $B_1(t_1)$ and $B_2(t_1)$.

§2.2 The Israel - Carter Conjecture

One very important area of investigation concerning black holes is the question of whether black holes are formed whenever a mass m is compressed until its circumference is less than or equal to $4\pi m$ in every direction, regardless of whether the collapse was spherical or non-spherical. A great deal of research has been done along these lines by Israel [14], Carter [3,4], and Hawking [12], among others. In this section, we will briefly summarize some of their conclusions.

Hawking [12] has considered spaces which satisfy the following conditions:

(i) They are weakly asymptotically simple;

(ii) They are stationary; i.e. there exists a one-parameter isometry group ϕ_t: $M \to M$ whose Killing vector K^α is timelike near $\mathscr{I}^+$ and $\mathscr{I}^-$;

(iii) There exist both a past event horizon $\partial I^+[\mathscr{I}^-]$ and a future event horizon $\partial I^-[\mathscr{I}^+]$;

(iv) There exists a partial Cauchy surface S from which the exterior region $\overline{I^+[\mathscr{I}^-] \cap I^-[\mathscr{I}^+]}$ can be determined. That is, the exterior region is contained in D(S);

(v) The two event horizons $\partial I^+[\mathscr{I}^-]$ and $\partial I^-[\mathscr{I}^+]$ intersect in a compact surface F.

Note that the existence of the past event horizon is to be understood as a condition on the analytic

continuation of the stationary solution to which the real solution tends. In a real situation where an initially nonsingular body collapses there will not be a past event horizon. However, Hawking has justified this condition by noting that one does not expect a fundamental difference between future and past in a stationary solution.

Hawking has proved that any rotating black hole satisfying the above five conditions must be axisymmetric, and he has then conjectured that any stationary black hole will be either static or axisymmetric. He has also proved that the surface F which is the intersection of the two horizons has the topology S^2.

A space-time M is said to be asymptotically Euclidean iff [18]

(i) There exists a compact $K \subset M$ and a diffeomorphism $\phi: M|K \to R^3|B$ where B is a closed ball centered at the origin;

(ii) With respect to the standard coordinate system in R^3, $\gamma_{ij} = \delta_{ij} + O(|x|^{-1})$, $\partial_K \gamma_{ij} = O(|x|^{-2})$, where $|x|^2 = \sum_{i=1}^{3} (x^i)^2$.

For stationary space-times, this implies Hawking's condition of asymptotic simplicity.

Israel [14] has shown that a space which

(i') is asymptotically Euclidean, and satisfies (ii), (iv) and (v) as above, must be the exterior Schwarzschild

space-time if it also satisfies the following conditions:

(vi) It is empty, i.e. $T_{\alpha\beta}=0$ in the exterior region;

(vii) It is static, and not merely stationary, i.e.the Killing vector K^α is hypersurface orthogonal, so $\eta^{\alpha\beta\gamma\delta}K_\beta K_{\gamma;\delta}=0$;

(viii) The gradient of $K^\alpha K_\alpha$ is not zero anywhere outside the horizon. This means that there is no neutral point at which a particle can remain at rest outside the black hole.

Carter [3,4] has shown that spaces which satisfy (i) to (vi) and

(ix) are axisymmetric,

(x) the surface F which is the intersection of the two horizons has the topology S^2,

fall into disjoint families, each depending on at least one and at most two independent parameters (one of which can be taken to be the asymptotically defined angular momentum h about the axis). One such family is known to exist, namely the Kerr spaces with $h \leqslant m^2$.

Wald [29,30] has shown that the analytic family of electrovac black hole metrics containing the Schwarzschild metric is completely spanned by the Kerr-Newman space-times with $a^2+e^2<m^2$. That is, the Kerr-Newman black holes are the only black hole solutions obtainable by analytic variation of the space-time geometry starting

from the Schwarzschild geometry.

These results taken together strongly suggest that every black hole settles down to a Kerr solution, assuming that everything becomes stationary eventually.

CHAPTER III

Brans-Dicke Theory: Predictions and Consequences

§3.1 Preliminaries

The field equations of the Brans-Dicke theory are:

$$\kappa T_{\alpha\beta} = \phi(R_{\alpha\beta} - 1/2\ Rg_{\alpha\beta}) - \omega\phi^{-1}(\phi_{/\alpha}\phi_{/\beta} - 1/2\ \phi^{/\gamma}\phi_{/\gamma}g_{\alpha\beta})$$

$$-\phi_{/\alpha\beta} + \Box\phi g_{\alpha\beta} \qquad ,$$

and

$$\Box\phi := \phi^{/\gamma}{}_{/\gamma} = (\kappa/(2\omega+3))T^{\gamma}{}_{\gamma}$$

where ω is a constant which has a suggested value of about 6, and which tends to ∞ in the limit of general relativity.

In this representation of the theory, test particles follow geodesics, and the gravitational "constant" G is only locally constant. However, for many purposes a different representation [8] of the theory is more convenient. Under the conformal transformation $\tilde{g}_{\alpha\beta}=\phi G_\circ g_{\alpha\beta}$, $\tilde{T}_{\alpha\beta}=\phi^{-1}G_\circ^{-1}T_{\alpha\beta}$ the field equations take the form

$$\kappa G_\circ\tilde{T}_{\alpha\beta}=\tilde{R}_{\alpha\beta} - 1/2\ \tilde{g}_{\alpha\beta}\tilde{R} - (3+2\omega)/(2G_\circ\phi^2)(\phi_{/\alpha}\phi_{/\beta} - 1/2\ \tilde{g}_{\alpha\beta}\tilde{g}^{\gamma\delta}\phi_{/\gamma}\phi_{/\delta})$$

$$\tilde{\Box}(\ln\phi) = \kappa/(3+2\omega)\ \tilde{T}^{\gamma}{}_{\gamma}$$

where differentiation is with respect to $\tilde{g}_{\alpha\beta}$.

In this frame, test particles do not follow geodesics, and the mass of the test particles is only locally constant, varying as $\phi^{-1/2}$, but the gravitational constant has a constant value G_o everywhere.

The weak-field solution for a stationary mass point of mass M, comparable to the weak-field solution in general relativity, is

$$\phi=\phi_o + 2M/(3+2\omega)c^2r \quad ,$$

$$g_{00}= -1 + (2M\phi_o{}^{-1}/rc^2)[1 + 1/(3+2\omega)] \quad ,$$

$$g_{ii}= 1 + (2M\phi_o{}^{-1}/rc^2)[1 - 1/(3+2\omega)] \quad , \; i=1,2,3$$

$$g_{\alpha\beta}/\gamma=0, \qquad \alpha\neq\beta$$

The spherically symmetric static solution is conveniently obtained by expressing the line element in isotropic form

$$ds^2= -e^{2\alpha}dt^2 + e^{2\beta} [dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)]$$

where α and β are functions of r only. For $\omega>3/2$, the general vacuum solution can be written in the form

$$e^{2\alpha} = e^{2\alpha^o} [(1-\beta/r)/(1+\beta/r)]^{2/\lambda}$$

$$e^{2\beta} = e^{2\beta^o} (1+\beta/r)^4[(1-\beta/r)/(1+\beta/r)]^{2(\lambda-c-1/\lambda)}$$

$$\phi = \phi_o[(1-\beta/r)/(1+\beta/r)]^{c/\lambda}$$

where $\lambda^2 = (c+1)^2 - c(1-\omega c/2) > 0$.

In case $\phi=$ constant, the vacuum scalar-tensor equations reduce to the vacuum Einstein equations, whose general

spherical solution is that of Schwarzschild.

§3.2 Predictions and Consequences

In considering any relativistic theory, it is natural to begin with its predictions concerning the three famous tests of general relativity, namely the gravitational red shift, the gravitational deflection of light, and the perihelion rotation of Mercury.

If the gravitational constant is defined as $G_o=\phi_o^{-1}(4+2\omega)(3+2\omega)^{-1}$, then the Brans-Dicke theory is in agreement with general relativity concerning the gravitational red shift.

The light deflection computed from the Brans-Dicke theory is

$$\delta\theta=(4G_oM/Rc^2)[(3+2\omega)/(4+2\omega)],$$

where R is the closest approach distance of the light ray to the sun of mass M. It differs from the general relativity value by the factor in brackets. However, the accuracy of light deflection observations is too poor to set any useful limit to the size of ω.

When the perihelion rotation rate of a planetary orbit is calculated using the above vacuum solution, the result is

$$[(4+3\omega)/(6+3\omega)] \times (\text{general relativity value}).$$

If it is assumed that the observed relativistic perihelion

rotation agrees with an accuracy of 8% or less with the computed result of general relativity, it is necessary to have $\omega \gtrsim 6$. For $\omega = 6$, the computed relativistic perihelion rotation rate is 39.4" arc / century. The value computed from general relativity is 42.89" arc / century. In the classical theory, 42.6" arc / century of the observational result was unaccounted for. However, measurements of the solar oblateness made by Dicke and Goldenberg [9] in 1966 seem to imply that 3.4" arc / century of the observed rate are accounted for by solar oblateness. After this is subtracted from the observed residual, there remains an effect, presumably relativistic, of 39.6" arc / century which lends support to the Brans-Dicke theory.

This result of Dicke and Goldenberg generated a great deal of interest in the Brans-Dicke theory, and a great many results were soon forthcoming.

Noerdlinger [21] proved that the constant ω must be non-negative. Toton [28] demonstrated that the Brans-Dicke theory is not a substantial improvement over general relativity with respect to Mach's Principle, since both theories disagree with Mach's Principle concerning solitary charged point particles and solitary neutral point particles.

Peters [25] geometrized the Brans-Dicke scalar field; Nutku [22] and Estabrook [10] derived the post-

Newtonian equations of hydrodynamics in the Brans-Dicke theory. Cocke and Cohen [6] showed that the Cauchy problem can be posed in a manner similar to general relativity.

Morganstern and Chiu [19] showed that, since scalar waves strongly damp the pulsation of neutron stars, the scalar field could be ruled out if a neutron star is found to pulsate.

Dehnen and Obregon [7] have shown that the cosmological solutions are in accordance with the usual data of age and mass-density of the universe.

O'Connell and Salmona [23] examined the radiation of gravitational waves in Brans-Dicke theory and showed that for all values of the eccentricity e of the orbit, the Brans-Dicke theory predicts that the rate of gravitational radiation from a system of binary stars is always smaller, by a factor of the order of 15/16 (where $\omega \simeq 6$), than the rate predicted by Einstein's theory.

Shapiro, et. al., [26] found that the experimentally measured bound on the time variation of the gravitational constant, $\dot{G}/G$, is below 3 parts in 10^{11}, which is nearly the value suggested by Dicke for the fractional yearly rate of change of G.

Sramek's [27] measured value for radiation deflection is in good agreement with Dicke's predicted value,

but is only 2 standard deviations from the general relativity prediction, so it is not a very strong result.

Dicke [5] has noted that the existence of scalar waves could cause variations in the moon's period, variations in earthquake rates, a general expansion of the earth, an adiabatic decrease in the internal temperature of the earth, and a decrease in the surface temperature of the earth. However, none of these effects can be reliably tested due to the complexity of the systems involved.

Thorne, Will, Ni [20], and their group at Cal. Tech. have recently been studying twentieth-century theories of gravity, and have called a theory viable if

(1) it is self-consistent,

(2) it is complete,

(3) it agrees, to within several standard deviations, with all experiments performed to date.

According to this definition, the Brans-Dicke theory has been found to be one of the few viable theories.

In light of the recent interest in black holes, it is not surprising that there has been speculation about black holes in Brans-Dicke theory. It was conjectured by Penrose and others that black holes in Brans-Dicke theory are precisely the same as in general relativity. This conjecture will be proven in Chapter IV for the static case.

CHAPTER IV

Static Black Holes in Brans-Dicke Theory

§4.1 Preliminaries

Having briefly discussed both black holes and the current status of the Brans-Dicke theory, we are now in a position to investigate black holes within the framework of the Brans-Dicke theory, keeping in mind the possibilities of testing general relativity which would be afforded if a qualitative difference between the two theories in this respect was detected.

In particular, we shall restrict our attention to static space-times, as defined in Chapter II, and the vacuum form of the Brans-Dicke field equations, as follows.

$$\phi(R_{\alpha\beta} - 1/2\ g_{\alpha\beta}R) = \omega\phi^{-1}(\phi_{/\alpha}\phi_{/\beta} - 1/2\ \phi^{/\gamma}\phi_{/\gamma}g_{\alpha\beta}) + \phi_{/\alpha\beta} + \Box\phi g_{\alpha\beta} \quad ,$$

and

$$\Box\phi := \phi^{/\gamma}{}_{/\gamma} = 0 \quad .$$

Greek indices run from 0 to 3, small Latin indices from 1 to 3, and capital Latin indices from 2 to 3; sign conventions are as in Landau, Lifshitz [16]. Stroke, comma, and semi-colon indicate differentiation with respect to 4-dim., 3-dim. and 2-dim. metrics respectively, and $\Box$,

Δ, and $\bar{\Delta}$ indicate the Laplacian in 4, 3, and 2 dimensions.

§4.2 Geometrical Equations

For a static space-time there exists a time-like, nowhere vanishing Killing vector field ξ^{α} which is hypersurface orthogonal. Let t be the global time defined by these hypersurfaces, and x^1, x^2, x^3 intrinsic coordinates of the hypersurface Σ. Then the metric can be written

$$ds^2=\gamma_{ij}(x^k)dx^i dx^j - V^2 dt^2 \quad ,$$

where $V^2 := -\xi^{\alpha}\xi_{\alpha}$

is a function on Σ, positive in the static region, which tends to 1 at infinity by the asymptotical flatness condition and to zero at the event horizon.

We assume that V has no critical points. Then V can be used as a coordinate in Σ, and the 3-metric written in the form

$$\gamma_{ij}dx^i dx^j=\bar{\gamma}_{AB}d\theta^A d\theta^B + \rho^2 dV^2 \quad ,$$

where $\bar{\gamma}_{AB}$ is the induced metric of a surface V=constant, and

$$\rho^{-2} := \gamma^{ij}V_{,i}V_{,j} \quad .$$

We also introduce the second fundamental form K_{AB} of the surfaces V=constant, where

$$K_{AB}= (\rho V_{,B})_{,A} \quad .$$

The following purely geometrical equations can now be obtained by calculating the Christoffel symbols and using

them to express the Riemann, Ricci, and Einstein tensors in 3-dim., and 2-dim. formalisms.

3-dimensional formalism

$$\overset{4}{R}_{00}=V\Delta V \quad , \qquad \overset{4}{R}_{i0}=0 \quad ,$$

$$\overset{4}{R}_{ij}= \overset{3}{R}_{ij} - V^{-1}V_{,ij} \quad ,$$

$$\overset{4}{R} = \overset{3}{R} - 2V^{-1}\Delta V \quad ,$$

(1) $$\overset{4}{R}_{\alpha\beta\gamma\delta}\overset{4}{R}^{\alpha\beta\gamma\delta}= 4V^{-2}V_{,ij}V^{,ij} + \overset{3}{R}_{ijkm}\overset{3}{R}^{ijkm} \quad ,$$

(2) $$\overset{3}{R}_{ijkm}\overset{3}{R}^{ijkm}= 4\overset{3}{G}_{ij}\overset{3}{G}^{ij} = 4\overset{3}{R}_{ij}\overset{3}{R}^{ij} - (\overset{3}{R})^{2} \quad .$$

2-dimensional formalism

(3) $$K_{AB}= 1/2\ \rho^{-1} \frac{\partial\overline{\gamma}_{AB}}{\partial V} \quad , \qquad \frac{\partial\sqrt{\overline{\gamma}}}{\partial V} = \sqrt{\overline{\gamma}}\rho K \quad ,$$

(4) $$\overset{3}{R}_{11}=\rho \frac{\partial K}{\partial V} + \rho^{2}K_{AB}K^{AB} + \rho\overline{\Delta}\rho \quad ,$$

$$\overset{3}{R}_{1B}=0 \quad ,$$

(5) $$\overset{3}{R}_{BC} = \rho^{-1} \frac{\partial K_{BC}}{\partial V} + \rho^{-1}\rho_{;BC} + \overline{R}_{BC} - KK_{BC} \quad ,$$

(6) $$\overset{3}{R} = 2\rho^{-1} \frac{\partial K}{\partial V} + 3\ K_{AB}K^{AB} + 2\rho^{-1}\overline{\Delta}\rho + \overline{R} - K^{2} \quad ,$$

$$\Delta V = -\rho^{-3} \frac{\partial\rho}{\partial V} + \rho^{-1}K \quad ,$$

(7) $$V^{-2}V_{,ij}V^{,ij} = (V\rho)^{-2}[K_{AB}K^{AB} + 2\rho^{-2}\rho_{;A}\rho^{;A} + \rho^{-4}\ (\partial\rho/\partial V)^{2}] \quad .$$

It should be emphasized at this point that no field equations were used in obtaining the above equations.

The following geometric equations are obtained for the scalar field ϕ of the Brans-Dicke theory in 3-dim. and 2-dim. formalism.

3-dimensional formalism

$$\phi_{/00} = -V^2 V^{,i}\phi_{,i} \quad , \quad \phi_{/0i} = 0 \quad , \quad \phi_{/ij} = \phi_{,ij}$$

$$\Box\phi = \Delta\phi + V^{-1}V^{,i}\phi_{,i}$$

2-dimensional formalism

$$\phi_{,11} = \frac{\partial^2\phi}{\partial V^2} - \rho^{-1}\frac{\partial\rho}{\partial V}\frac{\partial\phi}{\partial V} + \rho\rho^{;A}\phi_{;A}$$

$$\phi_{,1A} = \frac{\partial\phi_{;A}}{\partial V} - \rho^{-1}\rho_{;A}\frac{\partial\phi}{\partial V} - \rho K^B_A\phi_{;B}$$

$$\phi_{,AB} = \phi_{;AB} + \rho^{-1}K_{AB}\frac{\partial\phi}{\partial V}$$

$$\Delta\phi = \rho^{-2}\frac{\partial^2\phi}{\partial V^2} - \rho^{-3}\frac{\partial\rho}{\partial V}\frac{\partial\phi}{\partial V} + \rho^{-1}\rho^{;A}\phi_{;A} + \bar{\Delta}\phi + \rho^{-1}K\frac{\partial\phi}{\partial V}$$

where $V = x^1$.

§4.3 Field Equations

If we now introduce the Brans-Dicke field equations for the static vacuum case, we obtain the following equations in the 3-dimensional formalism.

$$(8) \qquad \left.\begin{aligned} \Delta\phi &= -V^{-1}V^{,r}\phi_{,r} \\ \phi V^{-1}\Delta V &= -V^{-1}V^{,r}\phi_{,r} \end{aligned}\right\} \quad \phi^{-1}\Delta\phi = V^{-1}\Delta V \quad ,$$

$$\phi\overset{3}{R} = \omega\phi^{-1}\phi_{,r}\phi^{,r} - 2V^{-1}V^{,r}\phi_{,r} \quad ,$$

$$\phi \overset{3}{R}_{ij} = \omega\phi^{-1}\phi_{,i}\phi_{,j} + \phi_{,ij} + \phi V^{-1} V_{,ij} \quad .$$

Therefore, in the 2-dimensional formalism, the above equations become

(9) $$K^{B}_{A;B} - K_{;A} = 0 \quad ,$$

(10) $$\bar{R} + K_{AB}K^{AB} - K^2 = \omega\phi^{-2}\phi_{;A}\phi^{;A} - $$
$$- \omega\phi^{-2}\rho^{-2}(\partial\phi/\partial V)^2 - 2\phi^{-1}V^{-1}\rho^{-2}\frac{\partial\phi}{\partial V}$$
$$- 2\phi^{-1}\rho^{-2}\frac{\partial^2\phi}{\partial V^2} + 2\phi^{-1}\rho^{-3}\frac{\partial\phi}{\partial V}\frac{\partial\rho}{\partial V}$$
$$+ 2V^{-1}\rho^{-3}\frac{\partial\rho}{\partial V} - 2\phi^{-1}\rho^{-1}\rho^{;A}\phi_{;A} \quad ,$$

(11) $$\rho^{-1}\frac{\partial K^{C}_{A}}{\partial V} + \rho^{-1}\rho^{;C}_{;A} + 1/2\ K_{BD}K^{BD}\delta^{C}_{A} + K(K^{C}_{A} - 1/2\ K\delta^{C}_{A})$$
$$= \omega\phi^{-1}(1/2\ \delta^{C}_{A}\phi_{;B}\phi^{;B} - \phi_{;A}\phi^{;C}) - \phi^{;C}_{;A}$$
$$- \phi V^{-1}\rho^{-1}K^{C}_{A} + 1/2\ \delta^{C}_{A}[\omega\phi^{-1}\rho^{-2}(\partial\phi/\partial V)^2$$
$$- 2V^{-1}\rho^{-2}(\partial\phi/\partial V)] - \rho^{-1}K^{C}_{A}\frac{\partial\phi}{\partial V} \quad ,$$

(12) $$\frac{\partial\rho}{\partial V} = \rho^2 K + \phi^{-1}\frac{\partial\phi}{\partial V} \quad .$$

Note that (12) together with (3) gives

(13) $$\frac{\partial}{\partial V}[\sqrt{\bar{\gamma}}/\rho] = -\sqrt{\bar{\gamma}}\phi^{-1}\rho^{-2}\frac{\partial\phi}{\partial V} \quad .$$

(14) $$\bar{\Delta}\phi = - V^{-1}\rho^{-2}\frac{\partial\phi}{\partial V} - \rho^{-2}\frac{\partial^2\phi}{\partial V^2} + \rho^{-3}\frac{\partial\rho}{\partial V}\frac{\partial\phi}{\partial V} - \rho^{-1}\rho^{;A}\phi_{;A}$$
$$- \rho^{-1}K\frac{\partial\phi}{\partial V} \quad ,$$

But (14) together with (3) gives

$$(15) \qquad \frac{\partial}{\partial V}\,[V\sqrt{\gamma}\rho^{-1}\,\frac{\partial\phi}{\partial V}] = -\,V(\sqrt{\gamma}\rho\phi^{;A})_{;A} \qquad .$$

§4.4 Proof that ϕ is constant

If space-time is asymptotically Euclidean, then for small $\varepsilon>0$ the surface $V = 1 - \varepsilon$ is diffeomorphic to a sphere, and therefore so are all surfaces V = constant, since V has no critical point. Moreover, an asymptotic expansion in terms of $1/r$ of equation (8) gives

$$(16) \qquad \begin{aligned} V &= 1 - m/r + O(r^{-2}) \quad , \\ \phi &= 1 + \alpha/r + O(r^{-2}) \quad , \end{aligned}$$

where ϕ is assumed to tend to 1 at infinity, m is a constant that can be shown to be the mass of the black hole, α is constant, and $r^2 = \gamma_{ij}x^i x^j$ for an asymptotically Cartesian coordinate system.

Now, if this static space-time did not contain a black hole, but was everywhere static and regular with some matter satisfying $T = T^{\alpha}_{\alpha} \leqslant 0$ (i.e. $0 \leqslant p \leqslant 1/3\ \varepsilon$ for a perfect fluid), then we have instead of (8)

$$\Delta\phi + V^{-1}V^{,i}{}_{,i} = \kappa T/(3 + 2\omega) \leqslant 0 \qquad ,$$

and the maximum principle implies that ϕ can have no minimum on Σ. If we assume that the static black hole was created by the collapse of such reasonable matter, we conclude that $\phi \geqslant 1$ everywhere in the static region. Since ϕ^{-1} stands for the gravitational constant, we may assume

that ϕ should remain bounded on the horizon.

Since the horizon is given by V=0, and is by assumption a regular surface of space-time, every 4-dimensional scalar is bounded. In particular, $\overset{4}{R}_{\alpha\beta\gamma\delta}\overset{4}{R}^{\alpha\beta\gamma\delta}$ is bounded. Therefore, (1) and (7) give

(17) $$1/4\ \overset{4}{R}_{\alpha\beta\gamma\delta}\overset{4}{R}^{\alpha\beta\gamma\delta} = 1/4\ \overset{3}{R}_{ijkm}\overset{3}{R}^{ijkm} + (V\rho)^{-2}[K_{AB}K^{AB} + 2\rho^{-2}\rho_{;A}\rho^{;A} + \rho^{-4}(\partial\rho/\partial V)^2] \quad ,$$

and (17) together with (12) gives

(18) $$K_{AB}(0,\theta^1,\theta^2) = O(V\rho) \quad , \quad K(0,\theta^1,\theta^2) = O(V\rho) \quad ,$$

(19) $$\rho_{;A}(0,\theta^1,\theta^2) = O(V\rho^2) \quad , \quad \frac{\partial\rho}{\partial V}(0,\theta^1,\theta^2) = O(V\rho^3) \quad ,$$

(20) $$\frac{\partial\phi}{\partial V}(0,\theta^1,\theta^2) = O(\phi V\rho^2) \quad .$$

But (13) and (20) give

$$\frac{\partial}{\partial V}[\sqrt{\gamma}/\rho]\ (0,\theta^1,\theta^2) = O(-\sqrt{\gamma}V) \quad .$$

Now, if the 2-surface V=0 has a regular induced metric, then ρ is bounded on V=0. Then (19) implies that $\rho(0,\theta^1,\theta^2)\equiv\rho_0$ =constant on the horizon.

Combining (10), (12), and (14) gives

$$V^{-1}K = 1/2\ \rho\bar{R} + 1/2\ \rho K_{AB}K^{AB} - 1/2\ \rho K^2 - 1/2\ \rho\omega\phi^{-2}\phi_{;A}\phi^{;A} + \phi^{-1}\rho^{-1}\frac{\partial^2\phi}{\partial V^2} - \phi^{-1}\rho^{-2}\frac{\partial\rho}{\partial V}\frac{\partial\phi}{\partial V} + 1/2\ \omega\phi^{-2}\rho^{-1}\frac{\partial\phi}{\partial V} + \phi^{-1}\phi_{;A}\rho^{;A} \quad .$$

Therefore, using (18), (19), and (20),

$$\text{(21)} \qquad \lim_{V\to 0} V^{-1}K = 1/2\ \rho_0\overline{R}(0,\theta^1,\theta^2) - 1/2\ \rho_0\omega\phi^{-2}\phi_{;A}\phi^{;A}(0,\theta^1,\theta^2) + \phi^{-1}\rho_0^{-1}\frac{\partial^2\phi}{\partial V^2}(0,\theta^1,\theta^2) + \phi^{-1}\rho^{;A}\phi_{;A}(0,\theta^1,\theta^2)$$

Contracting (11) and using (18), (19), (20) and (21) gives

$$\text{(22)} \qquad \lim_{V\to 0} V^{-1}K = 1/2\ \rho_0\overline{R}(0,\theta^1,\theta^2) - 1/2\ \rho_0\omega\phi^{-2}\phi_{;A}\phi^{;A}(0,\theta^1,\theta^2) + \frac{\partial K}{\partial V}(0,\theta^1,\theta^2) + V^{-1}\rho_0^{-2}\frac{\partial \rho}{\partial V}(0,\theta^1,\theta^2) \quad .$$

From (17) and (2), $4\ \overset{3}{R}_{ij}\overset{3}{R}^{ij} - (\overset{3}{R})^2$ is bounded. Recall (3), (4), (5), and (6), and note that

$$\overline{R}^{AB} = 1/2\ \overline{\gamma}^{AB}\overline{R} \quad \text{and} \quad \frac{\partial K}{\partial V} = \overline{\gamma}^{AB}\frac{\partial K_{AB}}{\partial V} - 2\rho K_{AB}K^{AB} \quad .$$

Then we obtain

$$\lim_{V\to 0} 1/4\ \overset{3}{G}_{ij}\overset{3}{G}^{ij} = \rho_0^{-2}\frac{\partial K_{BC}}{\partial V}\frac{\partial K^{BC}}{\partial V}(0,\theta^1,\theta^2) - 1/4\ (\overline{R})^2(0,\theta^1,\theta^2) \quad .$$

Recall that the 2-surface V=0 is assumed to have a regular induced metric. Then $\overline{R}(0,\theta^1,\theta^2)$ is bounded. This implies that $\frac{\partial K_{BC}}{\partial V}$ is bounded, which, with (18), implies that $\frac{\partial K}{\partial V}$ is bounded.

The preceding two statements, together with (18), (19), and (22), imply that $\phi_{;A}$ is bounded on the horizon. In fact, $\frac{\partial \phi}{\partial V}$ and $\phi_{;A}$ are bounded everywhere.

Integrate (15) over Σ, noting that the integral of the term on the right, being a 2-divergence, vanishes when taken over any closed 2-space V = constant. Thus, by Stokes's theorem,

$$\int\limits_{V=c}\int V\rho^{-1}\,\frac{\partial\phi}{\partial V}\,\sqrt{\bar{\gamma}}\,d\theta^1 d\theta^2$$

is independent of c. Since $\rho^{-1}\,\frac{\partial\phi}{\partial V}$ and the surface area are bounded on the horizon V=0, the integrand vanishes for c=0. Therefore, since ϕ itself is bounded on the horizon, it follows that

$$\int\limits_{V=0}\int V\phi\rho^{-1}\,\frac{\partial\phi}{\partial V}\,\sqrt{\bar{\gamma}}\,d\theta^1 d\theta^2 = 0 \quad . \tag{23}$$

From (15), the following identity can be obtained easily

$$(\sqrt{\bar{\gamma}})^{-1}\,\frac{\partial}{\partial V}\,[\sqrt{\bar{\gamma}}V\rho^{-1}\phi\,\frac{\partial\phi}{\partial V}] = V\rho^{-1}\,(\partial\phi/\partial V)^2 + V\rho\phi_{;A}\phi^{;A} - V(\sqrt{\bar{\gamma}})^{-1}(\phi\rho\sqrt{\bar{\gamma}}\,\phi^{;A})_{;A} \quad . \tag{24}$$

By integrating over Σ, the following inequality can be deduced

$$\int\limits_{V=1}\int (V\rho^{-1}\phi\,\frac{\partial\phi}{\partial V})\sqrt{\bar{\gamma}}\,d\theta^1 d\theta^2 \geq \int\limits_{V=0}\int (V\rho^{-1}\phi\,\frac{\partial\phi}{\partial V})\sqrt{\bar{\gamma}}\,d\theta^1 d\theta^2 = 0. \tag{25}$$

From (24) it is clear that equality holds iff

$$\frac{\partial\phi}{\partial V} = 0 \quad , \quad \phi_{;A} = 0 \tag{26}$$

everywhere on Σ. But the left side of (25) is zero by the asymptotic condition (16). Therefore, the fact that (26) must hold implies that ϕ = constant throughout Σ. Therefore, from (16), $\phi = 1$ in our units, and thus the field coincides with the Schwarzschild solution of Einstein's equations.

CHAPTER V

Conclusion

We have therefore proved in the static case the conjecture that black holes in Brans-Dicke theory are precisely the same as in general relativity.

It has recently been proved independently by Hawking [13] and by Bekenstein [1] that stationary black holes in Brans-Dicke theory are precisely the same as in general relativity.

Therefore, the results on stationary black holes in general relativity which were summarized in Chapter II are also valid in Brans-Dicke theory and there is no qualitative difference between the two theories regarding this phenomenon.

REFERENCES

[1] J. D. Bekenstein, Phys. Rev. D. 5 (1972) 2403.

[2] C. Brans and R. H. Dicke, Phys. Rev. 124 (1961) 925.

[3] B. Carter, J. Math. Phys. 10 (1969) 70.

[4] B. Carter, Phys. Rev. Letters 26 (1971) 331.

[5] "Gravitation and Relativity", Ed. H. Y. Chiu and W. F. Hoffmann, Benjamin (1963).

[6] W. J. Cocke and J. M. Cohen, J. Math. Phys. 9 (1968) 971.

[7] H. Dehnen and O. Obregon, Astrophys. and Space Sci. 15 (1971) 326.

[8] R. H. Dicke, Phys. Rev. 125 (1962) 2163.

[9] R. H. Dicke and H. M. Goldenberg, Phys. Rev. Letters 18 (1967) 313.

[10] F. B. Estabrook, Astrophys. J. 158 (1969) 81.

[11] R. Geroch, J. Math. Phys. 11 (1970) 437.

[12] S. W. Hawking, Commun. Math. Phys. 25 (1972) 152.

[13] S. W. Hawking, Commun. Math. Phys. 25 (1972) 167.

[14] W. Israel, Phys. Rev. 164 (1967) 1776.

[15] M. Johnson, Lett. al. Nuovo Cimento 4 (1972) 323.

[16] L. D. Landau and E. M. Lipshitz, "The Classical Theory of Fields", Oxford (1962).

[17] D. Lerner, in Lecture Notes in Physics 14, Ed. J. Ehlers, et. al., Springer-Verlag, Berlin (1972).

[18] A. Lichnerowicz, "Théories Relativistes de la Gravitation et de l'électromagnétisme", Masson, Paris (1955).

[19] R. E. Morganstern and H. Y. Chiu, Phys. Rev. 157 (1967) 1228.

[20] W. T. Ni, Astrophys. J. 176 (1972) 769.

[21] P. D. Noerdlinger, Phys. Rev. 170 (1968) 1175.

[22] Y. Nutku, Astrophys. J. 155 (1969) 999.

[23] R. F. O'Connell and A. Salmona, Phys. Rev. 160 (1967) 1108.

[24] R. Penrose, in Battelle Rencontres (1967), Ed. C. M. de Witt and J. A. Wheeler, Benjamin, New York (1968).

[25] P. C. Peters, J. Math. Phys. 10 (1969) 1029.

[26] I. I. Shapiro, et. al., Phys. Rev. Letters 26 (1971) 27.

[27] R. A. Sramek, Astrophys. J. Letters 167 (1971) 55.

[28] E. Toton, Phys. Rev. Letters 21 (1968) 1401.

[29] R. Wald, Phys. Rev. Letters 26 (1971) 1653.

[30] R. Wald, J. Math. Phys. 13 (1972) 490.